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Radiative corrections in symmetrized classical electrodynamics

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The physics of radiation reaction for a point charge is discussed within the context of classical electrodynamics. The fundamental equations of classical electrodynamics are first symmetrized to include magnetic charges: a double four-potential formalism is introduced, in terms of which the field tensor and its dual are employed to symmetrize Maxwell's equations and the Lorentz force equation in covariant form. Within this framework, the symmetrized Dirac-Lorentz equation is derived, including radiation reaction (self-force) for a particle possessing both electric and magnetic charge. The connection with electromagnetic duality is outlined, and an in-depth discussion of nonlocal four-momentum conservation for the wave-particle system is given.

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I. INTRODUCTION

Recently, close attention has been paid to the concept of duality in quantum-field-theories, as summarized by Witten [1]. In particular, recent work in superstring theory has resulted in the convergence of four main themes: electromagnetic duality in four dimensions, the symmetries of supergravity, dualities in superstring theory, and gauge theory dynamics in four dimensions.

The concept of duality in electrodynamics results from the symmetry between the electric and magnetic components of the field tensor: the source-free equations of the Maxwell set are symmetrical in vacuum under the transformation $\mathbf{E}\rightarrow\mathbf{B}$, and $\mathbf{B}\rightarrow-\mathbf{E}$; in addition, the symmetry can be maintained in the presence of four-currents, provided that both electric and magnetic monopoles are introduced, thus suggesting a deeper hidden symmetry. Since Dirac's brilliant insight on charge quantization [2–4], the role and importance of magnetic monopoles and duality in electrodynamics have taken on a much more profound significance. Feynman and Wheeler first demonstrated the deep connection between time-reversal and charge conjugation [5], while Schwinger proposed to associate the electric and magnetic charge in a single electrically charged monopole, referred to as a dyon [6]. Such a particle should exhibit the full symmetries of the electromagnetic interaction.

Although magnetic monopoles have never been observed, it can be argued that the apparent quantization of electric charge might represent indirect evidence for the existence of magnetic charge. The argument, originally put forth by Dirac [2,3] and later simplified by Saha [7], can be summarized as follows: if a magnetic field with nonzero divergence is added to an electric field with the same property, the total field has nonzero angular momentum, even in the static case. The field angular momentum turns out to be proportional to the product of the charges of the electric and magnetic sources involved, and independent of distance. Since angular momentum is quantized, and assuming that the amount of magnetic charge in the universe is finite, it follows that electric charge must be quantized. Although other explanations have been proposed for the observed quantization of charge, Dirac's argument remains the most elegant. In addition, in 1977, Montonen and Olive showed that in a limiting case of electroweak interaction theory, a particle of electric charge q and magnetic charge g, acquires a mass $m=\langle\psi\rangle\sqrt{q^2+g^2}$ under spontaneous symmetry breaking, where $\langle\psi\rangle$ is a constant measuring the gauge symmetry breaking [8].

Within this theoretical context, there exists a beautiful and compelling case for studying fully symmetrized versions of classical and quantum electrodynamics (CED and QED); in addition, these theories might also provide the correct approach to demonstrating that CED indeed represents the classical limit of QED, a problem that is still unresolved. This is because the duality of fully symmetrized QED implies that if the electric fine-structure constant $a=e^2/2\varepsilon_0 hc \cong 1/137.036$, and its magnetic counterpart $1/a$, are exchanged, electric and magnetic phenomena will appear to be switched, for a classical observer. This, in turn can be related to the notion of running coupling constants, used in gauge theory dynamics, where two important limiting cases might shed some light on the exact relation between QED and CED: the case where $a\rightarrow\infty$, and quantum effects disappear, and the case where the full symmetry between electricity and magnetism is restored, with $a=1$.

Thus, the main thrust of this paper is to present a classical derivation of radiation reaction for electric and magnetic monopoles, as well as dyons. The approach used here includes a generalization of Dirac's derivation of classical radiation reaction for a point charge from general principles, including gauge invariance and Lorentz covariance [9], where the double four-potential introduced by Cabbibo and Ferrari [10] is further extended by introducing a complex electromagnetic tensor unifying the conventional electromagnetic tensor and its dual. In addition, the electric and magnetic four currents are unified into a single complex four vector, and the connection with electromagnetic duality is now completely explicit: electric and magnetic charges can be rotated into one another, while preserving the global invariance of the symmetrized form of Maxwell's equations. One advantage of the complex four-potential formalism over

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Dirac's well-known model of magnetic monopoles is the absence of stringlike singularities; in addition, within this framework, one can readily derive a Hamiltonian for dyon-dyon interactions. Finally, our complex notation proves extremely compact, allowing for an elegant derivation of the symmetrized Dirac-Lorentz equation.

This paper is organized as follows: in Sec. II, we develop the aforementioned complex double four-potential formalism; the symmetrized form of the Dirac-Lorentz, equation, which describes the dynamics of a dyon with radiation reaction, is derived in Sec. III while the conceptual difficulties associated with this classical model for a point charge are briefly reviewed in Sec. IV, including electromagnetic mass renormalization, runaways, and acausal effects; finally, conclusions are drawn in Sec. V, where the implications of electromagnetic duality for QED are also outlined. In addition, a few technical points and a physical interpretation of the Schott term in terms of nonlocal four-momentum conservation, are presented in the Appendices, as well as a brief discussion of the Hamiltonian formalism for classical radiation reaction.

II. SYMMETRIZED ELECTRODYNAMICS

Here, and throughout the remainder of this paper, we use electron units, where length is measured in units of the classical electron radius $r_0=e^2/4\pi\varepsilon_0 m_0 c^2$, while time is measured in units of r_0/c, mass is measured in units of m_0, electric charge is measured in units of e, and magnetic charge in units of $\hbar/e$. In these units, $\varepsilon_0=1/4\pi$, $\mu_0=4\pi$, and for a particle of mass m_0, the four momentum is equal to the four velocity: $p_\mu=u_\mu=dx_\mu/d\tau$, where $x_\mu(\tau)$ is the world line of the particle, and τ is its proper time.

We now focus on the problem of a dyon, having both electric and magnetic charges q and g, respectively [6]. If magnetic sources are allowed, Maxwell's equations become symmetrized as follows:

$$\partial_\nu F^{\mu\nu}=4\pi j^\mu, \quad \partial_\nu \tilde{F}^{\mu\nu}=4\pi g^\mu, \tag{1}$$

where j^μ and g^μ correspond to the electric and magnetic four-current densities, and where $\tilde{F}^{\mu\nu}$ is the dual electromagnetic tensor. Here, the four-gradient operator [11,12] is defined as $\partial_\mu \equiv(-\partial_t,\nabla)$.

Now introducing the electric four-potential $A^\mu=(\phi,\mathbf{A})$, and its magnetic counterpart, $V^\mu=(\varphi,\mathbf{V})$, the field tensor and its dual may be written as follows:

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu-\varepsilon^{\mu\nu\alpha\beta}\partial_\alpha V_\beta \tag{2}$$

and

$$\tilde{F}^{\mu\nu}=\varepsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta+\partial^\mu V^\nu-\partial^\nu V^\mu, \tag{3}$$

where $\varepsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetrical Levi-Civita tensor [13]. Applying the Lorentz gauge condition to the four potentials, we have $\partial_\mu A^\mu=0$, and $\partial_\mu V^\mu=0$; and the symmetrized version of Maxwell's equations become

$$\partial_\nu\partial^\nu A_\mu=-4\pi j_\mu, \tag{4}$$

and

$$\partial_\nu\partial^\nu V_\mu=-4\pi g_\mu. \tag{5}$$

At this point, we note that Eqs. (4) and (5) are invariant under the dual transformation

$$A'_\mu=A_\mu\cos\theta+V_\mu\sin\theta, \quad V'_\mu=V_\mu\cos\theta-A_\mu\sin\theta, \tag{6}$$

provided that the charges and four currents are also transformed:

$$j'_\mu=j_\mu\cos\theta+g_\mu\sin\theta, \quad g'_\mu=g_\mu\cos\theta-j_\mu\sin\theta. \tag{7}$$

By analogy, with the idea that the Lorentz invariance becomes manifest in covariant notation, dual invariance can be clearly expressed in complex notation: since the dual transformation is essentially a rotation in the complex charge plane, this suggests the notation $\overline{q}=q+ig$, which yields the complex four-current density $\overline{j_\mu}=j_\mu+ig_\mu$; similarly, the complex four potential is defined as $\overline{A}_\mu=A_\mu+iV_\mu$, from which the complex electromagnetic field tensor is derived as

$$\overline{F}_{\mu\nu}=F_{\mu\nu}+i\tilde{F}_{\mu\nu}=\partial_\mu\overline{A}_\nu-\partial_\nu\overline{A}_\mu+i\varepsilon_{\mu\nu\alpha\beta}\partial^\alpha\overline{A}^\beta. \tag{8}$$

Within this context, the dual transform reduces to

$$\overline{F}'_{\mu\nu}=\overline{F}_{\mu\nu}e^{-i\theta}, \quad \overline{j}'_\mu=\overline{j}_\mu e^{-i\theta}. \tag{9}$$

This notation also proves extremely compact: the symmetrized form of Maxwell's equations takes the form

$$\partial_\mu\overline{F}^{\mu\nu}=-\partial_\nu\partial^\nu\overline{A}^\mu=4\pi\overline{j}^\mu, \tag{10}$$

and it is now obvious that Eqs. (8) and (10) are dual invariant.

Within this context, the dual-invariant Lorentz force equation takes the form

$$F_\mu=(qF_{\mu\nu}+g\tilde{F}_{\mu\nu})u^\nu=\mathrm{Re}(\overline{q}\overline{F}^*_{\mu\nu}u^\nu). \tag{11}$$

Note that whenever a product between any two complex electromagnetic quantities defined previously is taken, one of the quantities must be complex conjugated so that its magnetic component changes sign. This operation is analogous to raising and lowering an index in covariant notation, when contracting tensors, so that the sign of the timelike component is reversed: in both cases, the sign reversal ensures the invariance of the product under the respective transform. The duality and Lorentz transforms are both rotations, in two and four-dimensional spaces, respectively.

III. SYMMETRIZED DIRAC-LORENTZ EQUATION

To derive the radiation force, two different approaches can be used. First, one can follow Dirac's treatment [9,14] and derive the radiation reaction from general principles including gauge invariance and Lorentz covariance; this is the focus of the present section. The second type of derivation relies on a careful study of the conservation of the four momentum of the electromagnetic field [15–18] during the interaction.

In the case of a classical point dyon [6], which possesses both an electric charge q, and a magnetic charge g, the complex four current is given by

$$\overline{j}_\mu(x_\nu)=\overline{q}\int_{-\infty}^{+\infty}u_\mu(x'_\nu)\,\delta_4(x_\nu-x'_\nu)\,d\tau', \tag{12}$$

where the integral over proper time allows the use of the invariant four-dimensional Dirac delta function, as discussed in Appendix A.

Now assuming, as Dirac did, that a particle acts on itself through the Lorentz force [9], but using the symmetrized expression thereof, the self-force is

$$F_s^\mu=\mathrm{Re}[\overline{q}(\partial^\mu\overline{A}_s^\nu-\partial^\nu\overline{A}_s^\mu+i\varepsilon^{\mu\nu\alpha\beta}\partial_\alpha\overline{A}_\beta^s)^*u_\nu], \tag{13}$$

where the complex self-potential satisfies the driven wave equation

$$\partial_\nu\partial^\nu\overline{A}_\mu^s=-4\pi\overline{q}\int_{-\infty}^{+\infty}u_\mu(x'_\nu)\,\delta_4(x_\nu-x'_\nu)\,d\tau'. \tag{14}$$

Equation (13) can be written as

$$F_s^\mu=\mathrm{Re}[\overline{q}(\partial^\mu\overline{A}_s^{\nu*}-\partial^\nu\overline{A}_s^{\mu*})u_\nu-i\overline{q}\varepsilon^{\mu\nu\alpha\beta}\partial_\alpha\overline{A}_\beta^{s*}u_\nu], \tag{15}$$

while the driven wave equation (14) can be solved in terms of Green functions:

$$\overline{A}_\mu^s(x_\lambda)=-4\pi\overline{q}\int_{-\infty}^{+\infty}u_\mu(x'_\lambda)G(x_\lambda-x'_\lambda)\,d\tau', \tag{16}$$

where G is the Green function formally defined as $G(x_\lambda-x'_\lambda)\equiv-\delta_4(x_\lambda-x'_\lambda)/\partial_\nu\partial^\nu$. Using this Green function solution in Eq. (15), it becomes clear that the last term in the square brackets is pure imaginary: $-i\overline{q}\varepsilon^{\mu\nu\alpha\beta}\partial_\alpha\overline{A}_\beta^{s*}u_\nu\propto i\overline{q}\overline{q}^*=i|\overline{q}|^2$; thus, the self-force reduces to

$$F_s^\mu=\mathrm{Re}[\overline{q}(\partial^\mu\overline{A}_s^{\nu*}-\partial^\nu\overline{A}_s^{\mu*})u_\nu]. \tag{17}$$

Lest this manipulation appears as slight of hand, we depart momentarily from our elegant shorthand to elucidate the reason for the vanishing of the last term in Eq. (15): in terms of real quantities, the self-force reads

$$\begin{aligned}F_0^\mu&=q(\partial^\mu A_s^\nu-\partial^\nu A_s^\mu-\varepsilon^{\mu\nu\alpha\beta}\partial_\alpha V_p^s)u_\nu\\&\quad+g(\varepsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta^s+\partial^\mu V_s^\nu-\partial^\nu V_s^\mu)u_\nu,\end{aligned} \tag{18}$$

where the electric and magnetic self-potentials are driven by the dyon electric and magnetic four currents, with

$$\begin{bmatrix}A\\V\end{bmatrix}_\mu^s(x_\lambda)=-4\pi\begin{bmatrix}q\\g\end{bmatrix}\int_{-\infty}^{+\infty}u_\mu(x'_\lambda)G(x_\lambda-x'_\lambda)\,d\tau', \tag{19}$$

which implies that $V_\mu^s=(g/q)A_\mu^s$. This last relation allows some cancellation in Eq. (18), yielding the simpler expression

$$F_s^\mu=q(\partial^\mu A_s^\nu-\partial^\nu A_s^\mu)u_\nu+g(\partial^\mu V_s^\nu-\partial^\nu V_s^\mu-)u_\nu, \tag{20}$$

in agreement with Eq. (17).

Physically, the disappearance of the cross terms involving the action of the magnetic self-potential on the electric charge, and that of the electric self-potential on the magnetic charge, is due to the fact that the corresponding ponderomotive self-forces exactly cancel out. This decoupling of the radiation reaction forces is to be expected because the polarization of the radiation generated by the dyon's electric charge is always orthogonal to that radiated by the magnetic charge; thus, there is no interference between the electric and magnetic components of the dyon self-electromagnetic field.

Using the explicit form of the Green function in the force equation, we have

$$F_\mu^s(x_\lambda)=-4\pi|\overline{q}|^2\int_{-\infty}^{+\infty}u^\nu[u'_\nu\partial_\mu-u'_\mu\partial_\nu]G(x_\lambda-x'_\lambda)\,d\tau', \tag{21}$$

where we have used the notation $u'_\mu=u_\mu(x'_\lambda)$.

We now apply Dirac's procedure for finding the self-force in the point limit. The Green function in Eq. (21) depends on the space-time interval $s^2=(x-x')_\mu(x-x')^\mu$; using s^2 as the independent variable, the four-gradient operator reads $\partial_\mu\equiv2(x_\mu-x'_\mu)\partial/\partial s^2$, and the self-force is

$$F_\mu^s=-8\pi|\overline{q}|^2\int_{-\infty}^{+\infty}d\tau'\,u^\nu[u'_\nu(x_\mu-x'_\mu)-u'_\mu(x_\nu-x'_\nu)]\,\frac{\partial G}{\partial s^2}. \tag{22}$$

At this point, we introduce the new variable $\tau''=\tau-\tau'$, so that the range of integration explicitly includes the electron (singular point at $\tau''=0$). To evaluate the integral in Eq. (22), we can now use Taylor-McLaurin expansions in powers of τ'': we first have

$$\begin{aligned}x_\mu-x'_\mu&=x_\mu(\tau)-x_\mu(\tau-\tau'')\\&=\tau''u_\mu-\frac{1}{2}\tau''^2a_\mu+\frac{1}{6}\tau''^3\frac{da_\mu}{d\tau}+\cdots,\end{aligned} \tag{23}$$

where we have used the four-velocity and four-acceleration. For the four velocity, we have

$$u'_\mu=u_\mu(\tau-\tau'')=u_\mu-\tau''a_\mu+\frac{1}{2}\tau''^2\frac{da_\mu}{d\tau}+\cdots. \tag{24}$$

Using expansions (23) and (24), and factoring, we have

$$\begin{aligned}&\frac{u^\nu}{\tau''}[u'_\nu(x_\mu-x'_\mu)-u'_\mu(x_\nu-x'_\nu)]\\&\quad=(u^\nu u_\nu)\left(u_\mu-\frac{\tau''}{2}a_\mu+\frac{\tau''^2}{6}\frac{da_\mu}{d\tau}\right)+\frac{\tau''^2}{2}\left(u^\nu\frac{da_\nu}{d\tau}\right)u_\mu\\&\qquad-\left(u_\mu-\tau''a_\mu+\frac{\tau''^2}{2}\frac{da_\mu}{d\tau}\right)(u^\nu u_\nu)-\frac{\tau''^2}{6}\left(u^\nu\frac{da_\nu}{d\tau}\right)u_\mu\\&\qquad+O(\tau''^3).\end{aligned} \tag{25}$$

We now use the Lorentz invariant $u^\mu u_\mu=-1$. Differentiating this equation with respect to the proper time τ, we first find that $u_\mu(du^\mu/d\tau)=0=u_\mu a^\mu$; this result corresponds to the fact that the derivative of a vector with fixed length is orthogonal to the original vector: the four-acceleration is always perpendicular to the four velocity. Differentiating a second time with respect to τ, we also have $a_\mu a^\mu = -u_\mu(da^\mu/d\tau)$ [11,14,17].

Grouping terms, we finally obtain the important result

$$u^\nu[u'_\nu(x_\mu - x'_\mu) - u'_\mu(x_\nu - x'_\nu)] = \tau''^2\left\{-\frac{1}{2}a_\mu + \frac{\tau''}{3}\left[\frac{da_\mu}{d\tau} - u_\mu(a_\nu a^\nu)\right]\right\} + O(\tau''^4). \tag{26}$$

The relation between the space-time interval s^2 and the proper time difference τ'' can be expanded as well: $s^2 = \tau''^2(u_\mu u^\mu) - (\tau''^3/2)(u_\mu a^\mu + a_\mu u^\mu) + O(\tau''^4)$; using the orthogonality of the four velocity and four acceleration, this reduces to $s^2 = -\tau''^2 + O(\tau''^4)$, and we have $\partial G/\partial s^2 = [-1/2\tau'' + O(\tau'')]\partial G/\partial \tau''$.

With this, the expression for the self-force reads

$$F^s_\mu = 4\pi|\bar{q}|^2 \int_{-\infty}^{+\infty} d\tau'' \left\{ -(\tau''/2)a_\mu + \frac{\tau''^2}{3}\left[\frac{da_\mu}{d\tau} - u_\mu(a_\nu a^\nu)\right] + O(\tau''^3)\right\}\frac{\partial G}{\partial \tau''}. \tag{27}$$

We can integrate Eq. (27) by parts, according to $\int d\tau'' f(\tau'')\partial G/\partial\tau'' = -\int d\tau''(\partial f/\partial\tau'')G(\tau'')$, and obtain

$$F^s_\mu = -4\pi|\bar{q}|^2 \int_{-\infty}^{+\infty} d\tau'' \left\{ -\frac{1}{2}a_\mu + \frac{2}{3}\tau''\left[\frac{da_\mu}{d\tau} - u_\mu(a_\nu a^\nu)\right] + O(\tau''^2)\right\}G(\tau''). \tag{28}$$

We now use the retarded (causal) Green function [9,14]; Eq. (28) reads

$$F^s_\mu = |\bar{q}|^2 \int_{-\infty}^{+\infty} d\tau'' \left\{ -\frac{1}{2}a_\mu + \frac{2}{3}\tau''\left[\frac{da_\mu}{d\tau} - u_\mu(a_\nu a^\nu)\right] + O(\tau''^2)\right\}\frac{\delta(\tau'')}{|\tau''|}\left(1 + \frac{\tau''}{|\tau''|}\right), \tag{29}$$

where we have used $x_0 - x'_0/|x_0 - x'_0| = \tau''/|\tau''|$, and $\delta(s^2) = \delta(-\tau''^2) = \delta(\tau'')/|\tau''|$. This last identity has to be defined mathematically with care, as discussed in Appendix B.

We now proceed with the integration of Eq. (29) to obtain

$$F^s_\mu = -\frac{1}{2}|\bar{q}|^2\left[\int_{-\infty}^{+\infty}\frac{\delta(\tau'')}{|\tau''|}d\tau''\right]a_\mu + \frac{2}{3}|\bar{q}|^2\left[\frac{da_\mu}{d\tau} - u_\mu(a_\nu a^\nu)\right], \tag{30}$$

which is the sought-after expression for the self-force. Note that we have dropped the antisymmetrical terms in $\tau''/|\tau''|\delta(\tau'')$ and $(1/\tau'')\delta(\tau'')$, and that this expression is exact because all the higher-order terms in the expansion integrate out; this fact is rarely appreciated in the literature. The momentum-transfer equation, including the radiation reaction, now reads

$$\left[m + \frac{1}{2}|\bar{q}|^2\int_{-\infty}^{+\infty}\frac{\delta(\tau'')}{|\tau''|}d\tau''\right]a_\mu = \mathrm{Re}(\bar{q}\bar{F}^*_{\mu\nu}u^\nu) + \frac{2}{3}|\bar{q}|^2\left[\frac{da_\mu}{d\tau} - u_\mu(a_\nu a^\nu)\right]. \tag{31}$$

Equation (31) clearly exhibits the infinite electromagnetic mass, in the form of the divergent integral multiplying the four acceleration.

IV. CONCEPTUAL DIFFICULTIES: ELECTROMAGNETIC MASS RENORMALIZATION, RUNAWAYS, ACAUSAL EFFECTS

In this section, the main conceptual problems associated with the classical Dirac-Lorentz electron model are reviewed and discussed. The Dirac-Rohrlich asymptotic condition [9,16] is then introduced to determine the physical solutions of the Dirac-Lorentz equation.

As shown in Eq. (31), the mass term contains an infinite contribution from the self-electromagnetic fields of the point dyon. There are two different ways to circumvent this difficulty. First, we can consider that the infinite potential energy associated with a point-charge model must be balanced by an infinite binding energy $-W$, such as that produced by the Poincaré stress tensor [16,19,20], so that the finite observed rest mass of the dyon is given in units of m_0 by $m = (1/2)|\bar{q}|^2\int\delta(\tau'')/|\tau''|d\tau'' - W$. This procedure is essentially equivalent to mass renormalization in QED. The divergent electromagnetic mass, which is produced by the singular part of the Green function, can also be removed by considering the time-symmetrical Green function $G = (1/2)(G^- - G^+)$, as first proposed by Dirac [9]; here $G^\pm$ represent the retarded and advanced Green functions. There is little doubt that the removal of the infinite self-energy of the (nonradiative) Coulomb field is deeply connected to the charge conjugation and time-reversal properties of electrodynamics, as exemplified by the Wheeler-Feynman electrodynamics [21,22]; however, the connection is not entirely clear.

Using either approach to renormalize the electromagnetic mass, we finally obtain the complete equation of motion for a particle with arbitrary electric and magnetic charge:

$$ma_\mu = \mathrm{Re}(\bar{q}\bar{F}^*_{\mu\nu}u^\nu) + \frac{2}{3}|\bar{q}|^2\left[\frac{da_\mu}{d\tau} - u_\mu(a_\nu a^\nu)\right], \tag{32}$$

where m is the renormalized dyon mass. It is manifest that Eq. (32), like the generalized form of Maxwell's equations, is invariant under a duality transform. In the case of an electron, $\bar{q} = -1$, which yields the well-known Dirac-Lorentz equation:

$$a_\mu = -F_{\mu\nu}u^\nu + \tau_0\left[\frac{da_\mu}{d\tau} - u_\mu(a_\nu a^\nu)\right], \tag{33}$$

where $\tau_0 = 2/3$ is the Compton time scale, expressed in the units of r_0/c used here. In mksa units, $\tau_0 = \mu_0 e^2/6\pi m_0 c = 0.626\times 10^{-23}$ s. The first term on the right-hand side is the Lorentz force, while the radiation reaction contain the Schott term [9–12,14–20,23] and the radiation damping force [11,17–19,24].

A very important property of the Dirac-Lorentz equation is the fact that it satisfies energy-momentum conservation, as is easily seen by contracting Eq. (33) with the four velocity; we then have

$$u^\mu a_\mu = 0 = u^\mu F_{\mu\nu} u^\nu + \frac{2}{3}\left[u^\mu \frac{d a_\mu}{d\tau} - (u^\mu u_\mu)(a_\nu a^\nu)\right], \tag{34}$$

which is satisfied by virtue of the antisymmetry of the electromagnetic field tensor $F_{\mu v}$ and the orthogonality of u_μ and a_μ.

We now briefly review some of the conceptual difficulties associated with the Dirac-Lorentz equation itself. First, it is easily seen that, in the absence of an external field, Eq. (33) can be contracted with a^μ to obtain $a^\mu a_\mu = (\tau_0/2)(d/d\tau)(a^\mu a_\mu)$, which admits the so-called ''runaway'' solution $[a^\mu a_\mu](\tau) = [a^\mu a_\mu]_{\tau=0} \exp(2\tau/\tau_0)$.

Note that this self-excited motion implies that $[a^\mu a_\mu]_{\tau=0} \neq 0$, and can be eliminated through the use of the appropriate asymptotic conditions, $\lim_{\tau\to\pm\infty} a_\mu(\tau) = 0$, as suggested by Dirac [9] and Rohrlich [16]. This type of boundary condition on the electron motion also satisfies the law of inertia: the electron velocity remains constant when no external force is applied. A detailed analysis of Eq. (33) [16,20] also reveals the existence of acausal, or ''preacceleration'' solutions. This is directly connected to the implicit electromagnetic mass renormalization underlying the Dirac-Lorentz equation: the self-force can be explicitly derived by using the time-symmetrical Green function $G = (1/2)(G^- - G^+)$ [14], as first noted by Dirac [9]. As a result, although the electron is modeled as a point charge, it can interact electromagnetically with external fields localized within its classical radius: to show the implicit acausality of the Dirac-Rohrlich solution, we recast the Dirac-Lorentz equation in the form [16,20]

$$a_\mu - \tau_0 \frac{d a_\mu}{d\tau} = K_\mu \quad K_\mu = -F_{\mu\nu} u^\nu - \tau_0 u_\mu (a_\nu a^\nu). \tag{35}$$

Multiplication by the integrating factor $e^{-\tau/\tau_0}$ yields

$$\frac{d}{d\tau}[a_\mu(\tau) e^{-\tau/\tau_0}] = -\frac{1}{\tau_0} e^{-\tau/\tau_0} K_\mu(\tau); \tag{36}$$

Equation (36) can now be formally integrated to obtain

$$a_\mu(\tau) = \exp\left(\frac{\tau}{\tau_0}\right) \int_{-\infty}^{\tau} \exp\left(-\frac{\tau'}{\tau_0}\right) \times \left[\frac{1}{\tau_0} u^\nu F_{\mu\nu} + u_\mu (a_\nu a^\nu)\right](\tau') d\tau'. \tag{37}$$

The structure of this formal solution, which implicitly satisfies the Dirac-Rohrlich asymptotic condition, clearly exhibits the acausal convolution integral operator $\int_{-\infty}^{\tau} d\tau' \exp(-\tau'/\tau_0)$, which ''weights'' the externally applied electromagnetic field exponentially within a characteristic space-time interval equal to the classical electron radius. This type of solution does not run away because the preacceleration of the electron over the Compton time scale ''launches'' it on a stable trajectory. In other words, the preacceleration exactly compensates the runaway instability, and when the external field is applied, the electron executes a motion that conserves the total four momentum, including the pump and scattered fields, and asymptotically satisfies the law of inertia.

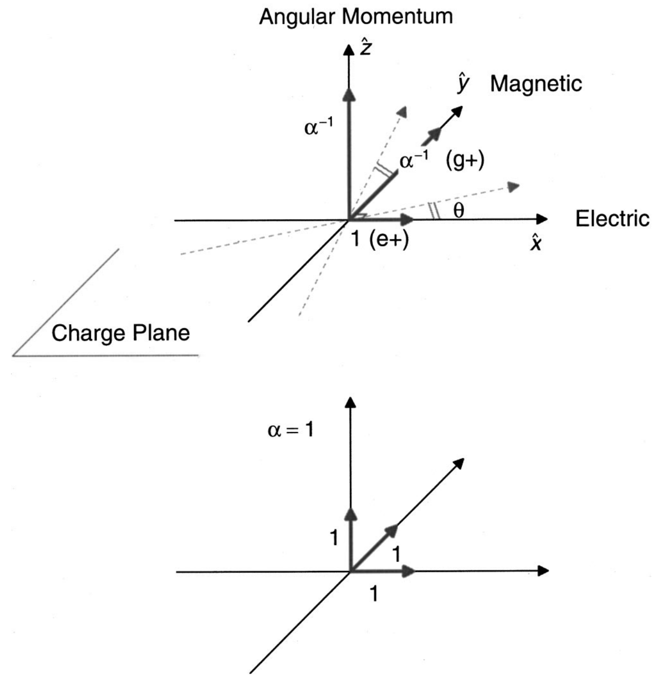

FIG. 1. Illustration of the Dirac-Schwinger quantization condition and the duality transform.

V. DISCUSSION

At this point, the connection between duality and the fully symmetrized version of electrodynamics can be discussed within the context of a dynamical gauge theory, where the fine structure constant is now a running coupling constant. We start from the Dirac-Schwinger charge quantization condition [6] for electric and magnetic monopoles:

$$\mathbf{q}_1 \times \mathbf{q}_2 = \mathrm{Im}(\overline{a_1}\overline{q_2^*})\hat{z} = n a^{-1}\hat{z}, \quad n \in \mathbb{N}. \tag{38}$$

In Eq. (38) the z axis corresponds to angular momentum; this is schematically illustrated on Fig. 1 (top) where two different charge state vectors are shown in the complex charge plane for a positron, with $\mathbf{q}_1 = [\hat{x}\mathrm{Re}(\overline{q_1}) + \hat{y}\mathrm{Im}(\overline{q_1})] = \hat{x}$, along the electric axis, and a magnetic monopole, $\mathbf{q}_2 = [\hat{x}\mathrm{Re}(\overline{q_2}) + \hat{y}\mathrm{Im}(\overline{q_2})] = a^{-1}\hat{y}$, along the magnetic axis. The $\pi/2$ angle between both charge states corresponds to the orthogonality of the electric and magnetic axes. The total angular momentum of the system is now represented by the cross product of $\mathbf{q}_1$ and $\mathbf{q}_2$, and is quantized according to Eq. (38). It is clear that a duality transform simply rotates the electric and magnetic axes, as shown in Fig. 1 (top); however, the cross product remains unchanged, as the relative angle between the monopole charge states and their length are preserved by this rotation. Therefore, to fully symmetrize electrodynamics, one needs to take $a=1$, as first observed by Dirac [2,3], in which case the distinction between electric and magnetic charges disappears. In this case, the radiation reaction are equal for an electric or a magnetic point charge interacting with external fields, and the full symmetry of electrodynamics is realized, as illustrated in Fig. 1 (bottom). One of the deepest questions associated with this theory is the exact connection with spin and the Dirac equation of QED [25].

In conclusion, the basic electrodynamic equations for a dyon have been presented within the context of a covariant formalism in the complex charge plane. A double-potential formalism has been introduced, which facilitates symmetrization of the calculations. An expression for the general self-force of a dyon has been derived, and it has been found that this expression is proportional to Dirac's expression for the self-force on an electron, differing only by a factor involving the electric and magnetic charge. Dirac's procedure for taking the point limit of the self-force has been applied, and the complete electrodynamic equation of motion for a dyon has been obtained. Finally, the connection with electromagnetic duality has been outlined.

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APPENDIX A: DYON FOUR-CURRENT

In Sec. II, the dyon four current is modeled by the integral over the dyon proper time of a four-dimensional delta function; here, we show how to go from a three-dimensional point charge model to an invariant delta function. In general, the four-current density can be expressed in terms of four-velocity and charge density as

$$j_{\mu}(x_{\lambda})=\left[\frac{u_{\mu}}{\gamma}\right](x_{\lambda})\rho(x_{\lambda}), \tag{A1}$$

which can be formally expressed as an integral over all times if we use the properties of the Dirac δ distribution:

$$j_{\mu}(x_{\lambda})=\int_{-\infty}^{+\infty}u_{\mu}(x'_{\lambda})\rho(x'_{\lambda})\,\delta(t-t')\,\frac{dt'}{\gamma'}. \tag{A2}$$

Here, $x'_{\lambda}=x_{\lambda}(t')\equiv[t',\mathbf{x}(t')]$, and is measured in units of r_0. The charge density of the dyon is now modeled by a three-dimensional δ distribution, and we have

$$j_{\mu}^{s}(x_{\lambda})=\overline{q}\int_{-\infty}^{+\infty}u_{\mu}(x'_{\lambda})\,\delta_3(\mathbf{x}-\mathbf{x}')\,\delta(t-t')\,d\tau', \tag{A3}$$

where we have introduced the dyon proper time, defined by $dt'=\gamma' d\tau'$. The invariant four-dimensional δ distribution can now be introduced, to yield

$$j_{\mu}^{s}(x_{\lambda})=\overline{q}\int_{-\infty}^{+\infty}u_{\mu}(x'_{\lambda})\,\delta_4(x_{\lambda}-x'_{\lambda})\,d\tau'. \tag{A4}$$

APPENDIX B

In Sec. III, we have used the identity:

$$\delta(s^2)=\delta(-\tau''^2)=\frac{\delta(\tau'')}{|\tau''|}. \tag{B1}$$

The identity given by Eq. (B1) has to be defined mathematically with care. We need to show that, for a certain class of suitably defined functions, we have

$$\int f(x)\,\delta(x^2)dx=\int f(x)\,\frac{\delta(x)}{|x|}\,dx. \tag{B2}$$

Starting from the well-known identity [14,19,26,27]

$$\delta(x^2-a^2)\equiv\frac{\delta(x-a)+\delta(x+a)}{2|a|}, \tag{B3}$$

and defining $g(x)=f(x)/|x|$, we first have

$$\begin{aligned}\int f(x)\,\delta(x^2-a^2)dx&=\int |x|g(x)\,\delta(x^2-a^2)dx\\&=\frac{|a|[g(a)+g(-a)]}{2|a|}\\&=\frac{g(a)+g(-a)}{2}.\end{aligned} \tag{B4}$$

Applying this result to a function $f(x)$ such that $\lim_{x\to 0}[f(x)/|x|]=g(0)$ exists, we can now write

$$\begin{aligned}\lim_{a\to 0}\left[\int f(x)\,\delta(x^2-a^2)dx\right]&=\lim_{a\to 0}\left[\frac{g(a)+g(-a)}{2}\right]\\&=g(0)\\&\equiv\int f(x)\,\delta(x^2)dx,\end{aligned} \tag{B5}$$

which is identical to

$$\int f(x)\,\frac{\delta(x)}{|x|}\,dx=\int g(x)\,\delta(x)dx=g(0). \tag{B6}$$

In this sense, the identity (B1) is properly defined.

APPENDIX C: SCHOTT TERM

Here, we consider the exchange of four-momentum between the electron, the external field, and the scattered field. An elementary treatment of this problem can be given in the instantaneous rest frame of the particle, as discussed by Jackson [19], where one can balance to zero the time-averaged work produced by the radiation force on the particle with the time-averaged radiated electromagnetic energy [19], to obtain the Schott term of the Abraham-Lorentz force [9,28–34]. The Schott term depends on the second time derivative of the particle velocity. However, it should be noted here that, strictly speaking, in the instantaneous rest frame ($\beta = \mathbf{0}$) where, by definition, both the particle velocity and kinetic energy are equal to zero, the infinitesimal variation of the work of the damping force, $dW=\mathbf{F}\beta d\tau$, must also be zero. In fact, it will be shown that in that frame, the dipole radiation pattern of the scattered field is symmetrical, and that there is no momentum exchanged between the charge and the radiated wave [17,18]. The method of derivation used here consists of evaluating the instantaneous variation of the energy momentum of the radiated field first [11,17,18]. This can be done either by integrating the Poynting vector flux and the radiation pressure of the scattered field on a sphere of finite radius, then taking the limit where the radius tends to zero, assuming no internal particle structure [18], or by generalizing results obtained in the instantaneous rest frame in a covariant way [11,20].

For a point charge moving along a world line $x_\mu(\tau)$ with three-velocity $\beta = d\mathbf{x}/dt$ and three-acceleration $\dot{\beta} = d\beta/dt$, the radiative electric field at r_μ is obtained by deriving the Liénard-Wiechert four potential. In electron units, we have for the radiative field [11,12,15–17,19,24,35–39]

$$\mathbf{E}(r_\mu) = -\left[\frac{\mathbf{n}\times(\mathbf{n}-\beta)\times\dot{\beta}}{(1-\beta\mathbf{n})^3 R}\right], \tag{C1}$$

where the quantities in the bracket are evaluated at the retarded time t^- such that $t-t^- = R(t^-) = |\mathbf{r}-\mathbf{x}(t^-)|$, and where $\mathbf{n}$ is the unit vector in the direction of observation.

The instantaneous electromagnetic momentum flux is given in terms of the Maxwell stress tensor [11,12,15–17,19,24,35–39], defined as

$$T_{ij} = \frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2)\delta_{ij}\right]. \tag{C2}$$

The total radiation pressure force applied to a sphere of radius R, corresponding to the momentum recoil of the photons emitted by the particle at t^- is given by $\int\int T_{ij}n_jR^2 d\Omega$, where n_j is the jth component of $\mathbf{n}$. Following Ref. [18], the instantaneous variation of the momentum of the scattered field can be expressed as

$$\frac{d\mathbf{G}}{dt} = -\lim_{R\to 0}\left[\int\int(\mathbf{n}\circ\mathbf{T})R^2 d\Omega\right], \tag{C3}$$

where $\circ$ denotes tensorial contraction.

The details of the derivation are given in Appendix D; the covariant form of the instantaneous variation of the scattered wave four momentum is found to be

$$\frac{dG_\mu}{d\tau} = \frac{2}{3}(a_\nu a^\nu)u_\mu \tag{C4}$$

The corresponding radiation damping force acting on the charge is essentially a relativistic effect. Indeed, if we first consider the instantaneous rest frame of the particle, we see that this force vanishes, as indicated by Eq. (C4). This is due to the symmetry of the dipole radiation pattern in this particular frame, as shown in Fig. 2 (top): although electromagnetic energy is radiated by the particle, there is no net recoil force because for each photon radiated in a given direction of space there is a photon with the same momentum radiated in the opposite direction. In any other frame, as shown in Fig. 2 (bottom), the relativistic Doppler effect breaks this symmetry: the photons radiated in the forward direction are blueshifted and carry more momentum than their backscattered counterparts, resulting in a net radiation force opposite to the direction of motion. In the instantaneous rest frame, the electron merely mediates the transfer of energy from the external field to the radiated wave by scattering the incident photons. This physical picture is in agreement with the fact that in that frame the electron has no free energy to yield, and that the work of any force acting on the electron must be zero; it also clearly indicates that in that frame, energy is directly exchanged between the external field and the scattered wave. With this in mind, we now need to carefully investigate the conservation of the energy momentum of the three interacting bodies. The covariant energy-momentum transfer equation between the charge and the electromagnetic field now takes the form $a_\mu = dp_\mu/d\tau = -F_{\mu\nu}u^\nu - dG_\mu/d\tau - dH_\mu/d\tau$, where the first term is the usual Lorentz force expressed in terms of the electromagnetic tensor, while the second term corresponds to the four-momentum radiated away by the scattered wave as derived above, and where we have introduced a third term corresponding to the instanta-

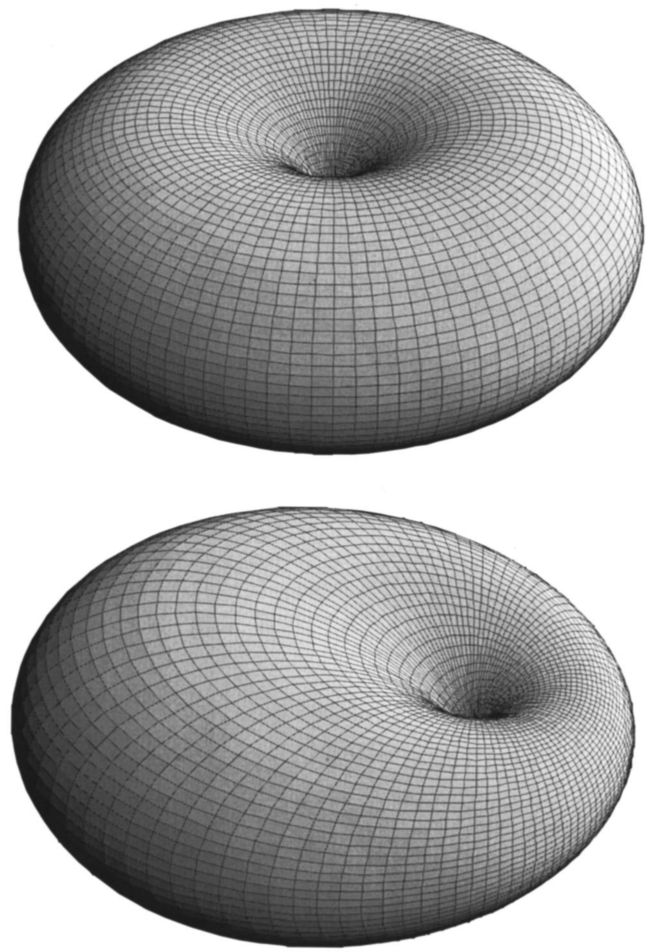

FIG. 2. Top: dipole radiation pattern, as observed in the instantaneous rest frame of the accelerated electron. Bottom: the same pattern, as observed in a frame where $\gamma = 1.01$.

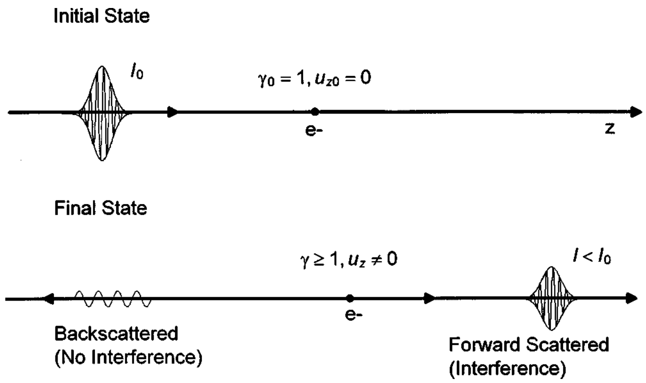

FIG. 3. Scattering of a laser pulse by an electron initially at rest.

neous variation of the energy momentum of the external field resulting from the interaction. Within this context, the radiation force is defined as $F^s_\mu = -d/d\tau(G_\mu + H_\mu)$; here, we have also used the principle of action and reaction, which holds as long as we consider the instantaneous interaction of a point particle: in that case, both the spacelike and timelike intervals are zero and there is no propagation delay to consider.

We now use the relations between the four-velocity and its successive derivatives [11,14,17]; using Eq. (C4), and contracting the four-momentum transfer equation with the four velocity, we first have

$$u^\mu a_\mu = 0 = -u^\mu F_{\mu\nu} u^\mu + \frac{2}{3}\left(u_\nu \frac{da^\nu}{d\tau}\right)(u^\mu u_\mu) - u^\mu \frac{dH_\mu}{d\tau}. \tag{C5}$$

The first term on the right-hand side is equal to zero, since the electromagnetic tensor is antisymmetrical; in the second term, we use $u^\mu u_\mu = -1$ to obtain $2/3 u_\nu da^\nu/d\tau = -u^\mu dH_\mu/d\tau$. As noted by Pauli [11], the general solution is $dH_\mu/d\tau = -(2/3) da_\mu/d\tau u_\mu + \kappa u^\nu K_{\mu\nu}$, where we have introduced the antisymmetrical tensor $K_{\mu\nu} = 2/3[u_\mu(da_\nu/d\tau) - u_\nu(da_\mu/d\tau)]$, and where κ is an arbitrary constant.

It is clear that $\kappa = 0$ yields the Dirac-Lorentz equation; in that case, we can identify the variation of the four momentum in the external field with the Schott term: $dH_\mu/d\tau = -2/3(da_\mu/d\tau)$. With this, the manifestly covariant expression for the radiation reaction becomes

$$F_\mu = \frac{2}{3}\left[\frac{da_\mu}{d\tau} - u_\mu(a_\nu a^\nu)\right], \tag{C6}$$

and it is easily seen that $F_\mu = u^\nu K_{\mu\nu}$. In addition, the antisymmetrical character of the tensor $K_{\mu\nu}$ guarantees that $u^\mu F_\mu = 0$. For completeness, we give the corresponding expression of the radiation reaction force in vector form, as expressed in electron units where the force is normalized to $m_0 c^2/r_0$:

$$\mathbf{F} = \tau_0 \gamma^2 \{\ddot{\beta} + 3\gamma^2 \dot{\beta}(\beta\cdot\dot{\beta}) + \gamma^2 \beta[\beta\cdot\ddot{\beta} + 3\gamma^2(\beta\cdot\dot{\beta})^2]\}. \tag{C7}$$

It is easily verified that the variation of the electron energy due to the radiative effects (timelike component of the radiation force) satisfies the equation

$$\frac{d\gamma}{dt} = \tau_0 \gamma^A [\beta\cdot\ddot{\beta} + 3\gamma^2(\beta\cdot\dot{\beta})^2] = \beta\cdot\mathbf{F}. \tag{C8}$$

Equation (C6) corresponds exactly to the covariant expression of the Abraham-Lorentz force. The self-interaction nature of the radiation force is evident, as the expression derived scales with the square of the particle charge: $\tau_0 = \mu_0 e^2/6\pi m_0 c$. In the first term of Eq. (C6), we recover the Schott term that depends on the second time derivative of the particle velocity, and which is identified here with the depletion of energy momentum from the pump (accelerating) field, while we recover the quadratic scaling with acceleration for the second term corresponding to the radiation damping force. As indicated by Eq. (C6), the total radiation force can be attributed to two distinct effects. On the one hand, energy momentum is radiated away by the scattered wave, as described by Eq. (C4). The asymmetry of the Doppler-shifted dipole radiation pattern in any frame where the particle is not instantaneously at rest, gives rise to this force, which dominates in the ultrarelativistic limit; it also has a nonzero value for a particle submitted to a constant acceleration, as opposed to the Schott term. On the other hand, the second term in Eq. (C6) is attributed to the energy momentum exchanged between the scattered wave and the external field. This term allows for the local simultaneous conservation of energy and momentum during the radiation process. The physics of the interaction can be illustrated by considering the process shown in Fig. 3. Here, we consider the total energy and momentum of the electrodynamical system initially comprising a high intensity, short-wavelength incoming laser pulse (pump) and an electron at rest. In general, after the interaction, the electron has gained some energy and momentum (in the minimal case, the electron would be left precisely at rest after the scattering), and is now moving at relativistic velocity, while the scattered wave carries energy and momentum in all spatial directions. In this case, it is clear that all the energy and momentum gained by both the electron and the scattered wave come at the expense of the external field. It is equally clear that in such a process, the radiated electromagnetic power and the variation of the electron energy cannot be equal, therefore invalidating any theoretical model based on the local conservation of four momentum between the electron and the radiated field only. We also note that while the backscattered radiation does not interfere with the laser pulse, the forward scattered radiation, which has the same spectral characteristics as the pump, and co-propagates in the positive z direction, does interfere destructively with the laser pulse and lowers its energy and momentum, yielding pump-field depletion.

Finally, in the case of an external electric field derived from a static potential $\varphi(\mathbf{r})$, the timelike component of the Dirac-Lorentz equation, which describes energy conservation, takes the simple form

$$\frac{d\gamma}{d\tau} = \mathbf{u}\cdot\nabla\varphi + \frac{2}{3}\frac{d^2\gamma}{d\tau^2} - \frac{dG_0}{d\tau} = \frac{d}{d\tau}\left[\varphi + \frac{2}{3}\frac{d\gamma}{d\tau} - G_0\right], \tag{C9}$$

and can formally be integrated to yield the conservation law $\Delta(\gamma - \varphi + G_0) = 2/3[d\gamma/d\tau]^{+\infty}_{-\infty}$, which indicates that, provided the Dirac-Rohrlich asymptotic condition $\lim_{\tau\to\pm\infty}[d\gamma/d\tau] = 0$ is satisfied, the electron potential energy is converted to kinetic energy and radiation.

Within this context, the small value of the fine-structure constant, which corresponds to the ratio of the classical to quantum electron scale (classical electron radius divided by the electron Compton wavelength), guarantees that the acausal effects related to the electromagnetic mass renormalization will be smeared by quantum fluctuations before the strong classical radiative correction regime is reached, thus preventing ‘‘naked acausalities.’’ If magnetic charges are considered, however, the radiation reaction dominate over the quantum effects because the effective coupling constant is now a^{-1}, which is a large number.

APPENDIX D: MAXWELL STRESS TENSOR

The instantaneous variation of the momentum of the scattered field can be expressed in terms of the electromagnetic stress tensor as

$$\frac{d\mathbf{G}}{dt}=-\lim_{R\to 0}\left[\int\int(\mathbf{n}\circ\mathbf{T})R^2 d\Omega\right], \tag{D1}$$

where $\circ$ denotes tensorial contraction.

Introducing the vector ξ, defined such that

$$\mathbf{E}=\frac{\dot{\beta}\xi}{(1-\mathbf{n}\cdot\beta)^3 R}, \tag{D2}$$

and using the fact that $\mathbf{B}=\mathbf{n}\times\mathbf{E}$, Eq. (D1) reduces to

$$\frac{dG_i}{dt}=\frac{1}{4\pi}\int\int n_j\left[\frac{\dot{\beta}^2\{\delta_{ij}\xi^2-\xi_i\xi_j-(n_j\xi_k-n_k\xi_j)(n_k\xi_i-n_i\xi_k)\}}{(1-\beta\cdot\mathbf{n})^5}\right]_{t=t^-} d\Omega. \tag{D3}$$

Following Sommerfeld [15], we change variables, and express the variation of momentum as a function of the retarded time. After some straightforward vector calculations, we obtain

$$\frac{d\mathbf{G}}{dt^-}=-\frac{1}{4\pi}\dot{\beta}^2\int\int\frac{[\xi\times(\xi\times\mathbf{n})]}{(1-\beta\cdot\mathbf{n})^5}d\Omega, \tag{D4}$$

which can be further reduced to

$$\frac{d\mathbf{G}}{dt^-}=\frac{1}{4\pi}\int\int\mathbf{n}\frac{\{\mathbf{n}\times[(\mathbf{n}-\beta)\times\dot{\beta}]\}^2}{(1-\beta\cdot\mathbf{n})^5}d\Omega, \tag{D5}$$

by noting that $\xi\times(\xi\times\mathbf{n})=(\mathbf{n}\cdot\xi)\xi-\xi^2\mathbf{n}$, and $\mathbf{n}\cdot\xi=0$. It is interesting to notice that Eq. (D5) can also be derived directly by using the Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ in the simpler equation $\lim_{R\to 0}(\int\int\mathbf{S}R^2 d\Omega)$, as shown in Ref. [18]. To evaluate the integral in Eq. (D5), we expand the numerator using spherical coordinates

$$\begin{aligned}\{\mathbf{n}\times[(\mathbf{n}-\beta)\times\dot{\beta}]\}^2 &=\dot{\beta}^2[(\sin\alpha\sin\theta\cos\phi+\cos a\cos\theta)^2(\beta^2-1)\\ &\quad+(1-\beta\cos\theta)^2+2\beta\cos a(1-\beta\cos\theta)\\ &\quad\times(\sin\alpha\sin\theta\cos\phi+\cos\alpha\cos\theta)].\end{aligned} \tag{D6}$$

Here, we have chosen the axis of the Galilean frame L such that we have

$$\beta=\hat{z}\beta,$$

$$\dot{\beta}=\dot{\beta}(\hat{z}\cos\alpha+\hat{x}\sin\alpha),$$

and

$$\mathbf{n}=\hat{x}(\sin\theta\cos\phi)+\hat{y}(\sin\phi\sin\phi)+\hat{z}\cos\theta.$$

The integral over all solid angles is

$$\frac{d\mathbf{G}}{dt^-}=\frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}\mathbf{n}\frac{\{\mathbf{n}\times[(\mathbf{n}-\beta)\times\dot{\beta}]\}^2}{(1-\beta\cdot\mathbf{n})^5}\sin\theta d\theta, \tag{D7}$$

where the explicit dependence of the numerator on θ and ϕ is given by Eq. (D6). The integral corresponding to the y component averages to zero over ϕ, and the integral corresponding to the x component averages to zero over θ. We are left with

$$\frac{d\mathbf{G}}{dt^-}=\hat{z}\frac{1}{4\pi}\dot{\beta}^2\left[\frac{8}{3}\pi\beta\gamma^6(1-\beta^2+\beta^2\cos^2\alpha)\right]. \tag{D8}$$

At this point it is important to note that, as the sphere radius tends to zero, the retarded time tends to the instantaneous interaction time; Eq. (D8) is easily shown to reduce to

$$\frac{d\mathbf{G}}{dt}=\frac{2}{3}\beta\gamma^4[\dot{\beta}^2+\gamma^2(\beta\cdot\dot{\beta})^2]. \tag{D9}$$

The instantaneous variation of the energy of the scattered wave can be derived in the same way by integrating the Poynting vector flux over all solid angles, and taking the limit where R tends to zero, to recover the Liénard formula

$$\frac{dW}{dt}=\frac{2}{3}\gamma^4[\dot{\beta}^2+\gamma^2(\beta\cdot\dot{\beta})^2]. \tag{D10}$$

The velocity-dependent term in Eqs. (D9) and (D10) can be expressed in terms of the four acceleration as

$$\gamma^4[\dot{\beta}^2+\gamma^2(\beta\cdot\dot{\beta})^2]=a_\mu a^\mu. \tag{D11}$$

The covariant generalization of Eqs. (D9) and (D10) then becomes quite straightforward. Following Becker [17], we combine Eqs. (D9) and (D10) to obtain the sought-after covariant form of the instantaneous variation of the energy momentum of the scattered wave:

$$\frac{dG_\mu}{d\tau}=\frac{dG_\mu}{dt}\frac{dt}{d\tau}=\gamma\frac{dG_\mu}{dt}=\frac{2}{3}(a_\nu a^\nu)u_\mu. \quad \text{(D12)}$$

APPENDIX E: HAMILTONIAN FORMALISM

It is also quite instructive to consider the dynamics of a point electron within the context of a Hamiltonian formalism. It can be shown that, in the temporal gauge, the Hamiltonian

$$H=-\sqrt{(\pi-\mathbf{A})^2+\mu^2}+\frac{1}{16\pi}\int\int\int(\mathbf{E}^2+\mathbf{B}^2)d^3\mathbf{x}, \quad \text{(E1)}$$

yields the covariant Dirac-Lorentz equation in an external field $F_{\mu v}$ provided that the mass term in Eq. (A32) satisfies the condition

$$\mu=\frac{1}{2}\int\frac{\delta(\tau)}{|\tau|}d\tau-1, \quad \text{(E2)}$$

which corresponds to the mass renormalization previously introduced. In Eq. (E1), π is the particle canonical momentum, and $\mathbf{A}(q)$ is the vector potential at the position of the particle. Note that in Eq. (E1), there is a negative sign in front of the term usually associated with the kinetic energy. This readily explains the existence of runaway solutions. It is equally important to notice that the normalized electron mass in the Dirac-Lorentz equation has the usual value of one, and not the value of μ given in the Hamiltonian [Eq. (E1)]. In this sense, it is clear that the (dissipative) Dirac-Lorentz equation cannot be derived from a conventional Hamiltonian (as expected).

One should also recognize that, by neglecting the radiation reaction terms in the Dirac-Lorentz equation, which then reduces to the usual Lorentz force equation, one can recover the standard Hamiltonian in an external potential,

$$H=\sqrt{(\pi-\mathbf{A})^2+1}+\varphi, \quad \text{(E3)}$$

which defines a positive particle kinetic energy, as exemplified by the plus sign in front of the square root.

Furthermore, in the case where the external force is derived from a potential, it is possible to integrate the timelike component of the Dirac-Lorentz equation to obtain

$$\Delta\gamma=\Delta\varphi-W+\frac{2}{3}\left[\frac{d\gamma}{d\tau}\right]_{-\infty}^{+\infty}, \quad \text{(E4)}$$

where the usual balance between the kinetic, potential, and radiated energy is realized, as long as $\lim_{\tau\to\infty}[d\gamma/d\tau]=\lim_{\tau\to\infty}[d\gamma/d\tau]$, which the Dirac-Rohrlich asymptotic condition obviously satisfies. Because of the implied nonlocality of the radiation, this balance is generally not realized differentially.

Finally, it is worth noting that the runaway solutions of the Dirac-Lorentz equation correspond to (unphysical) trajectories that minimize the particle energy associated with the Hamiltonian given in Eq. (E1) by making it tend to negative infinity. In this sense, there is an interesting analogy between the Dirac-Rohrlich asymptotic conditions (law of inertia), which are assumed by Dirac to yield the only physical solution to the Dirac-Lorentz equation [9], and the assumption made in QED that the negative-energy states are entirely occupied by electrons in order to prevent transitions from positive to negative energies. The general problem of the classical limit of QED remains an outstanding difficulty in classical electrodynamics at high-field strengths. For example, using the path-integral formulation of QED, the photon coordinates can be functionally integrated out [40], but the radiation reaction terms yield divergences on the electron world line that are exactly analogous to those discussed in Sec. II. However, the Dirac-Lorentz equation, coupled to the prescription that all runaway solutions must be excluded, offers a simple and economical classical electron model that yields a consistent electrodynamics that includes the usual Maxwell-Lorentz theory and gives a reasonable description of such phenomena as nonlinear Compton scattering, which can now be studied experimentally at energies in the 50 GeV range and laser intensities exceeding 10^{18} W/cm^2 [41,42]. Finally, we note that the Hamiltonian formalism can also be generalized within the framework of symmetrized electrodynamics: the symmetrized Hamiltonian including radiation reaction is simply given by

$$\begin{aligned}H=&-\sqrt{(\pi-e\mathbf{A}-g\mathbf{V})^2+\mu^2}+e\phi+g\varphi\\&+\frac{1}{8\pi}\int\int\int\left(\frac{E^2+B^2}{2}-\mathbf{V}\cdot\nabla\times\mathbf{E}+\mathbf{A}\cdot\nabla\right.\\&\left.\times\mathbf{B}-\varphi\nabla\cdot\mathbf{E}-\phi\nabla\cdot\mathbf{B}\right)d^3\mathbf{x},\end{aligned} \quad \text{(E5)}$$

while the Lorentz force Hamiltonian is $H=\sqrt{(\pi-e\mathbf{A}-g\mathbf{V})^2+m^2}+e\phi+g\varphi$. Of course, there is no Hamiltonian for the Dirac-Lorentz equations following Eq. (E5).

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